Two-dimensional Ultra-Toda integrable mappings and chains (Abelian case)

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Abstract

The new class of integrable mappings and chains is introduced. Corresponding $(1+2)$ integrable systems invariant with respect to such discrete transformations are represented in explicit form. Soliton like solutions of them are represented in terms of matrix elements of fundamental representations of semisimple A_n algebras for a given group element.

1 Introduction: the key role of Toda chains in the theory of integrable systems

There are many different forms for representation of infinite Toda chain. The most known and useful are the following ones

$$(\ln v)_{xy} = \frac{\overleftarrow{v}}{v} - \frac{v}{\overrightarrow{v}}, \quad (\ln \theta)_{xy} = \overleftarrow{\theta} - 2\theta + \overrightarrow{\theta} \tag{1}$$

where $v \equiv v_s$, $\overleftarrow{v} \equiv v_{s+1}$, $\overrightarrow{v} \equiv v_{s-1}$, s natural number and x, y are independent coordinates of the problem.

Equations (1) may be considered as definition of some mapping: the law by help of which two initial functions $v, \overrightarrow{v}$ $(\theta, \overrightarrow{\theta})$ are associated with two final ones $\overleftarrow{v}, v$ $(\overleftarrow{\theta}, \theta)$.

We restrict ourself by the first system (1) and rewrite it in the following equivalent form

$$\overleftarrow{u} = \frac{1}{v}, \quad \overleftarrow{v} = v(vu - (\ln v)_{xy}) \tag{2}$$

The remarkable property of the mapping (substitution) (2) consists in its integrability [1],[2]. This mean that corresponding to it symmetry equation (arising as variation derivative of the substitution by itself)

$$\overleftarrow{U} = -\frac{1}{v^2}V, \quad \overleftarrow{V} = v^2U + (2uv - (\ln v)_{xy})V - v(\frac{V}{v})_{xy} \tag{3}$$

possess the sequence of nontrivial solutions [3]. In (3) it suggests that independent arguments of functions U, V are u, v (from (2)) and its derivatives on space coordinates up to the definite order wile $\overleftarrow{U}, \overleftarrow{V}$ are the same functions arguments of which are shifted by help of (2).

Each solution of (3) may be connected with the integrable in Liouville sense (infinite number of conservations "laws" in involution) system of evolution type equations

$$u_t = U, \quad v_t = V \tag{4}$$

Moreover the last systems are invariant with respect to substitution (2).

The most part of integrable systems and equations resolved up to now by different methods (particular by inverse scattering one) are directly connected with Toda symmetry (2) (or its numerous auto-Backlund transformations) in described above scheme [4].

Under appropriate boundary conditions infinite Toda chain is interrupted and over go into integrable finite dimensional system different classes solution of which it is possible to represent in explicit form. The most known ones are Toda chain with fixed ends $v_0^{-1} = v_N = 0$ or periodical Toda $v_0 = V_N$. In the first case it is possible to find general solution [5] (depending on necessary number of arbitrary functions). In the second case the parametric soliton-like subclass of solutions [6]. The general solution in this case may be represented in the form of infinite absolutely convergent series [7].

General solution of Toda chain with fixed ends in the case $N = 2m$ directly is connected with m-soliton solution of evolution type equations (4). Namely u_m, v_m of Toda chain with fixed ends is exactly m-soliton solution (under some additional restrictions on arbitrary functions) of evolution type equations (4).

So we see that equations of Toda chain place ambivalent role: it define the form of integrable systems (as solution of its symmetry equation) and its interrupting version give possibility to find different classes solutions of such systems in explicit form.

In papers [1],[2] it was assumed that the theory of integrable systems is equivalent to the representation theory of the group of integrable substitutions with respect to which the Toda chain system is the simplest partial case.

In the present paper we introduce the new class of integrable mappings and chains. These chains are different from the usual Toda one by more number of unknown functions in each point of the lattice. These mappings are integrable and it is possible to construct the hierarchy of integrable systems each one of which is invariant with respect to such discrete substitution. General solution for the case of fixed ends may be represented in terms of matrix elements of fundamental representations of semisimple A_n algebras (groups).

2 Integrable chains connected with the graded algebras

In the paper [8] (these results are literally repeated in corresponding chapters of monograph [9]) it was proposed a general method for construction of exactly integrable systems connected with arbitrary graded (super) algebras. We will use it below for the case of A_n semisimple seria.

As usual by $X^{\pm}_{\alpha}, h_{\alpha}$ we denote the generators of simple roots together with corresponding Cartan elements. The $\pm s$ graded subspaces consist from generators of A_n algebra which can be constructed from the commutators of s simple roots. The general equation of [8]

$$[\partial_x - \sum_{s=1}^{m_1} A^{-s}, \partial_y - (\rho h) - \sum_{s=1}^{m_2} A^{+s}] = 0 \tag{5}$$

in the case under consideration may be concretized more detail to represent integrable chains in more observable form. Generators of $\pm s$ graded subspaces are the following ones

$$Y^{+s}_{\alpha} = [X^{+}_{\alpha} ... [X^{+}_{\alpha+s-1}, X^{+}_{\alpha+s}]..], \quad Y^{-s}_{\alpha} = [[..[X^{-}_{\alpha+s}, X^{-}_{\alpha+s-1}] .... X^{-}_{\alpha}] \tag{6}$$

with obvious commutation relations as a corollary of their definition:

$$
\begin{aligned}
[Y^{-i}_\alpha, Y^{+j}_\beta] &= \delta_{\alpha+i,\beta+j} Y^{-i+j}_\alpha - \delta_{\alpha,\beta} Y^{-i+j}_{\alpha+j}, \quad i \le j \\
[Y^{+i}_\alpha, Y^{+j}_\beta] &= -\delta_{\alpha,\beta+j} Y^{i+j}_\alpha + \delta_{\alpha+i,\beta} Y^{i+j}_\beta \qquad (7) \\
[Y^{-i}_\alpha, Y^{+i}_\beta] &= \delta_{\alpha,\beta} \sum_{s=0}^{i-1} h_{\alpha+i}
\end{aligned}
$$

Further

$$A^{-s} = \sum_\alpha Y^{-s}_\alpha f^s_\alpha, \quad A^{+s} = \sum_\beta Y^{+s}_\beta \bar{f}^s_\beta$$

The arising as a consequence of (5) system of equations takes the form

$$
\begin{aligned}
&(f^s_\alpha)_y + (\sum_{k=0}^{s-1} \bar{\rho}_{\alpha+k}) f^s_\alpha - \sum_{k=1}^{m_1-s} (f^{s+k}_\alpha \bar{f}^k_{\alpha+s} - f^{s+k}_{\alpha-k} \bar{f}^k_{\alpha-k}) = 0 \\
&\bar{\rho}_\alpha = -\rho_{\alpha+1} + 2\rho_\alpha - \rho_{\alpha-1}, \quad (\rho_\alpha)_x + \sum_{s=1}^{min(m_1,m_2)} \sum_{k=0}^{s-1} f^s_{\alpha-k} \bar{f}^s_{\alpha-k} = 0 \qquad (8) \\
&(\bar{f}^s_\alpha)_x - \sum_{k=1}^{m_2-s} (\bar{f}^{s+k}_\alpha f^k_{\alpha+s} - \bar{f}^{s+k}_{\alpha-k} f^k_{\alpha-k}) = 0
\end{aligned}
$$

We call (8) as UToda(m_1, m_2) chain keeping in mind that usual two-dimensional Toda system for A_n seria corresponds to the choice $m_1 = m_2 = 1$. The rezults of [8] give garantie that system (8) is exactly integrable and give prescription how to integrate it.

3 UToda (2,2) system and its general solution

In this case in each point of the chain there are 5 unknown functions $\rho_\alpha, f^1_\alpha, \bar{f}^1_\alpha, f^2_\alpha, \bar{f}^2_\alpha$. But due to the gauge invariance not all of them are independent and after introduction gauge invariant values $q_\alpha = f^1_\alpha \bar{f}^1_\alpha, p_\alpha = \frac{f^2_\alpha}{f^1_\alpha f^1_{\alpha+1}}, \bar{p}_\alpha = \frac{\bar{f}^2_\alpha}{f^1_\alpha f^1_{\alpha+1}}$ we rewrite (8) in the chain form with three independent functions in each point

$$(\ln p)_y + (\overleftarrow{q}^2 \overleftarrow{p} - qp) + (\overleftarrow{q} p - \overrightarrow{q} \overrightarrow{p}) = 0$$

$$-(\ln q)_{xy} + (\overleftarrow{q}\bar{p} - \overrightarrow{q}\overrightarrow{\bar{p}})_y + (\overleftarrow{q}p - \overrightarrow{q}\overrightarrow{p})_x + \hat{K}(q + \overrightarrow{q}q\overrightarrow{p}\overrightarrow{\bar{p}} + q\overleftarrow{q}p\bar{p}) = 0 \qquad (9)$$

$$(\ln \bar{p})_x + (\overleftarrow{q}^2\overleftarrow{\bar{p}} - q\bar{p}) + (\overleftarrow{q}\bar{p} - \overrightarrow{q}\overrightarrow{\bar{p}}) = 0$$

where $\hat{K}\theta = \overleftarrow{\theta} - 2\theta + \overrightarrow{\theta}$.

In the case $p = \bar{p} = 0$ we come back to usual Toda chain system (UToda (1,1)). The case $\bar{p} = 0$ (or equivalent to it $p = 0$) corresponds to UToda (1,2) chain with equations

$$-(\ln q)_{xy} + (\overleftarrow{q}p - \overrightarrow{q}\overrightarrow{p})_x + \overleftarrow{q} - 2q + \overrightarrow{q} = 0$$

$$(10)$$

$$(\ln p)_y + (\overleftarrow{q}^2\overleftarrow{p} - qp) + (\overleftarrow{q}p - \overrightarrow{q}\overrightarrow{p}) = 0$$

The last system is interesting by itself, but always can be considered as reduction of UToda (2,2) chain under definite choice of arbitrary functions defined it's solution.

Below we represent the general solution of system (9) after rewriting it in more suitable and observable form.

Let us use the following substitutions

$$p = \exp(\overleftarrow{s} + s), \quad \bar{p} = \exp(\overleftarrow{t} + t), \quad \theta = q\exp(s + t)$$

In new variables system (9) takes the form

$$(\exp -s)_y = \theta\,\overleftarrow{\exp} -t - \theta\,\overrightarrow{\exp} -t, \quad (\exp -t)_x = \theta\,\overleftarrow{\exp} -s - \theta\,\overrightarrow{\exp} -s$$

$$-(\ln\theta)_{xy} + \hat{K}(\theta\exp -(s+t) + \theta\overleftarrow{\theta} + \theta\overrightarrow{\theta}) = 0$$

or finally after identification $p^{(1} = \exp -s, \bar{p}^{(1} = \exp -t$

$$(p^{(1)})_y = \overleftarrow{\bar{p}^{(1}}\theta - \overrightarrow{\bar{p}^{(1}}\theta, \quad (\bar{p}^{(1)})_x = \overleftarrow{p^{(1}}\theta - \overrightarrow{p^{(1}}\theta$$

$$(11)$$

$$-(\ln\theta)_{xy} + \hat{K}(p^{(1}\bar{p}^{(1}\theta + \theta\overleftarrow{\theta} + \theta\overrightarrow{\theta}) = 0$$

Anyone can agree that the last form is more attractive compare with (9), although both are equivalent to each other (at least in the case of interrupted chain).

To represent the general solution of (9) or (11) for us it will be necessary the knowledge of some facts from [8]. Here we reproduce them in conspective form. Two equations of S-matrix type are in foundation of the whole construction

$$(M_+)_y = M_+L_+ \equiv M_-(\sum_1^r Y^{+1}_\alpha \bar\phi^1_\alpha + \sum_1^{r-1} Y^{+2}_\beta \bar\phi^2_\beta)$$

$$(M_-)_x = M_-L_- \equiv M_-(\sum_1^r X^-_\alpha \phi^1_\alpha + \sum_1^{r-1} Y^{-2}_\beta \phi^2_\beta) \tag{12}$$

where r is the rank of semisimple algebra, $Y^{\pm 2}$ are defined by (8).

The "Lagrangian" functions $L^\pm$ of equations (12) are correspondingly upper and lower triangular matrixes and by these reasons solutions of (12) may be represented in quadratures.

The solution of the problem may be expressed via matrix elements of the following A_n group element

$$K = \exp(h\Phi)M_-^{-1}M_+ \exp-(h\bar\Phi) \equiv m_-^{-1}m_+ \tag{13}$$

As it follows from its definition groups elements $m_\pm$ satisfy the equations:

$$(m_+)_y = m_+(-(h\bar\Phi)_y + \sum Y^{+1}_j(\bar\nu_{j+1}-\bar\nu_{j-1}) + \sum Y^{+2}_j \equiv$$
$$m_+(\exp(h\bar\Phi)L_+\exp-(h\bar\Phi) - (h\bar\Phi)_y)$$

$$(m_-)_x = m_-((h\Phi)_x + \sum Y^{-1}_j(\nu_{j+1}-\nu_{j-1}) + \sum Y^{-2}_j \equiv$$
$$m_-(\exp(h\Phi)L_-\exp-(h\Phi) - (h\Phi)_x) \tag{14}$$

The last equalities determine all introduced above values and relations between them.

By $\| i\rangle, (\langle i \|)$ we will denote the minimal vector of i-th fundamental representation of A_n algebra with the properties

$$X^-_\alpha \| i\rangle = 0, \quad h_s \| i\rangle = -\delta_{s,i}, \quad \langle i \| X^+_\alpha = 0, \quad \langle i \| h_s = -\delta s, i \tag{15}$$

$$X_\alpha^+ \parallel i\rangle = \delta_{\alpha,i} X_i^+ \parallel i\rangle, \quad \langle i \parallel X_\alpha^- = \delta_{\alpha,i} \langle i \parallel X_i^-$$

The following abbreviations will be used throughout the whole paper

$$[i] = \langle i \parallel K \parallel i\rangle, \quad \theta_i = \frac{[i+1][i-1]}{[i]^2}$$

$$\alpha_{ij..l} \equiv \frac{\langle i \parallel X_i^- X_j^- ... X_l^- K \parallel i\rangle}{\langle i \parallel K \parallel i\rangle}, \quad \bar{\alpha}_{ij..l} \equiv \frac{\langle i \parallel K X_l^+ .... X_j^+ X_i^+ \parallel i\rangle}{\langle i \parallel K \parallel i\rangle}$$

In these notations the general solution of the system (11) may be represented in the form

$$\bar{p}_i^{(1} = (\bar{\nu}_{i+1} - \bar{\alpha}_{i+1} - \bar{\nu}_{i-1} + \bar{\alpha}_{i-1} \equiv (\theta_i)^{-1} (\alpha_i)_y$$

$$\theta_i = \frac{[i+1][i-1]}{[i]^2} \tag{16}$$

$$\bar{p}_i^{(1} = (\nu_{i+1} - \alpha_{i+1} - \nu_{i-1} + \alpha_{i-1} \equiv (\theta_i)^{-1} (\bar{\alpha}_i)_x$$

In"old" variables solution of (9) may be expressed via (16) as

$$q_i = p_i^{(1} \bar{p}_i^{(1} \theta_i, \quad \bar{p}_i = \frac{1}{\bar{p}_i^{(1} \bar{p}_{i+1}^{(1}}, \quad p_i = \frac{1}{p_i^{(1} p_{i+1}^{(1}}$$

For checking of the validity of represented above solution only one relation between matrix elements of different fundamental representations is necessary. Namely

$$Det_2 \begin{pmatrix} \langle i \parallel K \parallel i\rangle & \langle i \parallel X_i^- K \parallel i\rangle \\ \langle i \parallel K X_i^+ \parallel i\rangle & \langle i \parallel X_i^- K X_i^+ \parallel i\rangle \end{pmatrix} = [i+1][i-1] \tag{17}$$

In fact (17) is nevertheless then famous Yakoby equality connecting the determinants of $i, i+1, i-1$ orders rewritten in the more economical form [5].

The proving of this relation reader can find in [10] (see also appendix). All other necessary relations are direct corollary of the last one.

By help of these relations it is not difficult to prove that

$$(\ln[i])_{xy} = \theta_i \theta_{i+1} + \theta_i \theta_{i-1} + \bar{p}_i^{(1} p_i^{(1} \theta_i$$

and other equalities of the same kind (partially containing in equations of equivalence (16)). The details of corresponding calculations reader can find in section 5.

Solution of UToda(1,2) chain contains among constructed above. By the same kind of transformation as over go from (9) to (11) we obtain instead of (10)

$$\phi_y = \overleftarrow{\theta} - \overrightarrow{\theta}, \quad (\ln\theta)_{xy} = \overleftarrow{\phi}\theta - 2\phi\theta + \overrightarrow{\phi}\theta \tag{18}$$

For general solution of the last chain we have: ϕ coincides with $p_i^{(1}$ of (16) and in the expression for θ (16) it is necessary a little modification:

$$\theta_i = \nu_i \bar{\phi}_i^1 \frac{\langle i-1 \parallel K \parallel i-1\rangle\langle i+1 \parallel K \parallel i+1\rangle}{(\langle i \parallel K \parallel i\rangle)^2}$$

Of course in equation determining $M_\pm$ (12) it is necessary put $\bar{\phi}_i^2 = 0$.

From explicit form of solution (16) we see that it defines by only one group element K and so it is possible to wait that all problems connected with UToda chains systems may be resolved on the level of its properties.

4 Parameters of evolution - Hamiltonians flows

In this section we introduce the parameters of evolution and represent the way of construction the systems of equations invariant with respect to UToda substitutions. We begin the discussion from the simplest case of usual Toda chain for which solution of the problem is known [3].

4.1 Two-dimensional Davey-Stewartson hierarchy

In this case the group element m_+ is defined by equation

$$m'_+ \equiv (m_+)_y = m_+(-(h\bar{\Phi})' + \sum Y_j^{+1} \tag{19}$$

and depend on the set of arbitrary functions $\bar{\Phi}_i$. Let try to find the last functions in such a way that equation

$$\dot{m}_+ \equiv (m_+)_{t_2} = m_+(-(h\dot{\bar{\Phi}}) + \sum Y_j^{+1}\mu_j - \sum Y_j^{+2}) \tag{20}$$

would be selfconsistent with (19). Maurer-Cartan identity after its trivial resolution takes the following form

$$\mu_i = -\bar\Phi_{i+1} + \bar\Phi_{i-1}, \quad \mu_i' + (k\dot{\bar\Phi})_i - \mu_i (k\bar\Phi)' = 0 \tag{21}$$

where as usually $(kf)_i = -f_{i+1} + 2f_i - f_{i-1}$. Finally (21) is equivalent to

$$\dot{\bar\Phi}_i - \dot{\bar\Phi}_{i-1} + \bar\Phi_i'' + \bar\Phi_{i-1}'' + (\bar\Phi_i' - \bar\Phi_{i-1}')^2 = 0 \tag{22}$$

(in all cases from the condition $b_i = b_{i+1}$ we have done conclusion $b_i = 0$).

Solution of the chain system (22) it is possible to express via the N (N-is the number of the points of interrupted chain) linear independent solutions of the single one-dimensional Schrodinger equation

$$\dot\Psi = \Psi'' + V(t_2, y)\Psi$$

where V is arbitrary function of space y and time t_2 coordinates [3].

The chain (ref17) induced the Davey-Stewartson system [11]. To explain this fact let us consider matrix element $[i]$ and calculate its derivatives with respect to arguments y, t_2 (we use notations introduced in (15) and below). As a consequence of (19),(20) and (22) we have

$$\begin{gathered} \dot{\ln[i]} = \dot{\bar\Phi}_i + \mu_i \bar\alpha_i - \bar\alpha_{i,i+1} + \bar\alpha_{i,i-1} \\ (\ln[i])' = (\bar\Phi_i)' + \bar\alpha_i \\ \frac{[i]''}{[i]} = (\bar\Phi_i)'' + (\bar\Phi_i')^2 + (\bar\Phi_{i+1} + \bar\Phi_{i-1})'\bar\alpha_i + \bar\alpha_{i,i+1} + \bar\alpha_{i,i-1} \end{gathered} \tag{23}$$

Excluding $\dot{\bar\Phi}_i, \bar\Phi_i', \bar\Phi_i''$ from (22) by help of (23) we come to the key equality

$$\ln \dot{\frac{[i]}{[i-1]}} + (\ln([i][i-1])'' + ((\ln \frac{[i]}{[i-1]}{}')^2 = 2(\bar\alpha_{i-1,i} + \bar\alpha_{i,i-1} - \bar\alpha_{i-1}\bar\alpha_i) \tag{24}$$

remarkable by the fact that its right-hand side is identically equal to zero due to recurrent relations between the matrix elements of different representations of A_n groups (see appendix).

Introducing the functions $v = \frac{[i]}{[i-1]}, u = \frac{[i-2]}{[i-1]}$, bearing in mind the main equation of Toda chain by itself

$$(\ln[i])_{xy} = \frac{[i+1][i-1]}{[i]^2}$$

and the fact that equality (24) is correct for arbitrary i, we conclude that functions u, v satisfy the following system of equations

$$-\dot{u} + u_{yy} + 2u \int dx(uv)_y = 0 \quad \dot{v} + v_{yy} + 2v \int dx(uv)_y = 0 \tag{25}$$

This is exactly Davey-Stewartson system [11]. In one-dimensional limit - usual nonlinear Schrodinger equation.

In general case equation (20) defined algebra valued function $m_+^{-1}\dot{m}_+$ it is necessary to change on the condition that this function is decomposed on generators of algebra the graded index of which is less then some given natural number say r. In this case we will obtain system of equation which determine dependence $\bar{\Phi}_i$ on parameter $\bar{t}_r$ and obtain the corresponding system of equations of two-dimensional D-S hierarchy. By different method this problem in explicit form was solved in [3], [10].

The construction described above in one-dimensional case equivalent to multi-time formalism and corresponding technique of Hamiltonian flows [4].

Of course all done above it is possible to repeat with respect space coordinate x in the pair with group element m_-.

As a result we will obtain the sequence of right $\bar{t}_s$ and left t_l evolution parameters, corresponding system of equation invariant with respect to Toda discrete substitution and it's particular explicit multi-soliton type solutions.

4.2 PToda(2.2) case

Now we want to apply the technique of the last subsection for construction of unknown up to now example of integrable system invariant with respect UToda(2,2) substitution (11). We omitted as a rule the calculations by themselves, representing only the finally results. All necessary formulae for it's independent verification reader find in section 5 and Appendix. These calculation are not difficult but very long in consequent rewriting (may be because of the bad notations or very straightforward attempts to realize them by known for us methods).

In this case element m_+ satisfy the equation (see section 3):

$$(m_+)' = m_+(-(h\bar{\Phi})' + \sum Y_j^{+1}\bar{\phi}_j + \sum Y_j^{+2} \equiv$$

$$m_+(\exp(h\bar{\Phi})L_+ \exp -(h\bar{\Phi}) - (h\bar{\Phi})') \quad f' \equiv f_y, \quad \bar{\phi}_j = (\bar{\nu}_{j+1} - \bar{\nu}_{j-1})$$

Corresponding operator of t_2 differentiation has the form

$$\dot{m}_+ = m_+((-(h\dot{\bar{\Phi}}) + \sum Y_j^{+1}\mu_j^{(1} + \sum Y_j^{+2}\mu_j^{(2} + \sum Y_j^{+3}\mu_j^{(3} - \sum Y_j^{+4}) \quad (26)$$

Condition of selfconsistency (Maurer-Cartan identity) gives possibility to express all functions $\mu_i^{(s}$ from (26) in terms of $\bar{\Phi}_i, \bar{\nu}_i$ and find the system of equations which the last functions as functions of y, t_2 arguments satisfy.

By help of commutation relations (7) all calculations are straightforward. Below reader can find result of them:

$$\mu_i^{(3} = \bar{\phi}_{i+2} + \bar{\phi}_i = \bar{\nu}_{i+3} - \bar{\nu}_{i-1} \quad \mu_i^{(2} = \bar{\Phi}'_{i+1} - \bar{\Phi}'_{i+2} - \bar{\Phi}'_i + \bar{\Phi}'_{i-1} + \bar{\phi}_i\bar{\phi}_{i+1}$$

$$\mu_i^{(1} = -(\bar{\nu}_{i+1} - \bar{\nu}_{i-1})(\bar{\Phi}'_{i+1} - \bar{\Phi}'_{i-1}) - (\bar{\nu}'_{i+1} + \bar{\nu}'_{i-1})$$

The chain system of equation with respect to unknown functions $\bar{\Phi}, \bar{\nu}$ (compare with (22) in this case has the form:

$$\dot{\bar{\Phi}}_{i+1} - \dot{\bar{\Phi}}_{i-1} + \bar{\Phi}''_{i+1} + \bar{\Phi}''_{i-1} + (\bar{\Phi}'_i - \bar{\Phi}'_{i+1})^2 + (\bar{\Phi}'_i - \bar{\Phi}'_{i-1})^2 = 2\bar{\nu}'_i(\bar{\nu}_{i+1} - \bar{\nu}_{i-1})$$

$$(\bar{\nu}_{i+1} \dot{-} \bar{\nu}_{i-1}) + (\bar{\nu}_{i+1} + \bar{\nu}_{i-1})'' - 2\bar{\nu}'_{i+1}(\bar{\Phi}'_i - \bar{\Phi}'_{i-1}) - 2\bar{\nu}'_{i-1}(\bar{\Phi}'_i - \bar{\Phi}'_{i+1}) +$$
$$(\bar{\nu}_{i+1} - \bar{\nu}_{i-1}) \times \quad (27)$$
$$(\dot{\bar{\Phi}}_{i+1} - 2\dot{\bar{\Phi}}_i) + \dot{\bar{\Phi}}_{i-1} + \bar{\Phi}''_{i+1} - \bar{\Phi}''_{i-1} + (\bar{\Phi}'_{i+1})^2 - (\bar{\Phi}'_{i-1})^2 - 2(\bar{\Phi}'_{i+1} - \bar{\Phi}'_{i-1})\bar{\Phi}'_i = 0$$

In what follows in this section we deviate from introduced in the last section notations and consider UToda$(2,2)$ in variables

$$p_i = \alpha_i - \nu_i, \quad \bar{p}_i = \bar{\alpha}_i - \bar{\nu}_i$$

In this variables as it follows from (16) UToda$(2,2)$ substitution takes the form

$$(p_i)_y = \theta_i(\bar{p}_{i-1} - \bar{p}_{i+1}), \quad (\bar{p}_i)_x = \theta_i(p_{i-1} - p_{i+1})$$

$$(\ln\theta)_{xy} = \hat{K}(p_y\bar{p}_x\theta + \theta\overleftarrow{\theta} + \theta\overrightarrow{\theta}) \quad (28)$$

Let us define functions $v_i = \frac{[i+1]}{[i]}, u_i = \frac{[i-1]}{[i]}$. Obviously $\theta_i \equiv u_i v_i, u_{i+1} = v_i^{-1}$.

The following equalities are direct corollary of all introduced above definitions and may be verified directly (all necessary formulae reader can find in section 5 and in Appendix)

$$\frac{\dot{v}_i}{v_i} + \frac{v''_i}{v_i} + 2(\bar{\alpha}_{i,i-1} - \bar{\nu}_{i-1}\bar{\alpha}_i)' - 2\bar{p}'_i\bar{p}_{i+1} = V_i(y, t_2)$$

$$-\frac{\dot{u}_i}{u_i}+\frac{u''_i}{u_i}-2(\bar{\alpha}_{i,i+1}-\bar{\nu}_{i+1}\bar{\alpha}_i)'+2\bar{p}'_i\bar{p}_{i-1}=U_i(y,t_2)$$
$$\dot{p}_i=-p''_i-2(\ln v_{i-1})'+2\theta_i\bar{p}'_{i-1}$$
$$V_i=\dot{\bar{\Phi}}_{i+1}-\dot{\bar{\Phi}}_i+\bar{\Phi}''_{i+1}-\bar{\Phi}''_i+(\bar{\Phi}'_i-\bar{\Phi}'_{i+1})^2-2\bar{\nu}'_i\bar{\nu}_{i+1}$$
$$U_i=-\dot{\bar{\Phi}}_{i-1}+\dot{\bar{\Phi}}_i+\bar{\Phi}''_{i-1}-\bar{\Phi}''_i+(\bar{\Phi}'_i-\bar{\Phi}'_{i-1})^2+2\bar{\nu}'_i\bar{\nu}_{i-1}$$

To have some closed system it is necessary to exclude from the last system of equalities the terms containing functions $\bar{\alpha}_{i\pm1},\bar{\alpha}_{i,i\pm1}$. The following additional equalities

$$(\bar{\alpha}_{i,i\pm1}-\bar{\nu}_{i\pm1}\bar{\alpha}_i)_x=\mp\theta_i\theta_{i\pm1}+\theta_i\bar{p}_{i\pm1}(\bar{p}_i)_x$$

solve this problem. After this keeping in mind equations of substitution (28) it is possible to rewrite the previous system of equalities in the closed form for 8 "unknown" functions $v_{i+1},v_i,u_i,u_{i-1},p_i,p_{i+1},\bar{p}_i,\bar{p}_{i-1}$

$$\dot{v}_{i+1}+v''_{i+1}+2v_i\bar{p}'_{i+1}p'_{i+1}+2v_{i+1}\bar{p}_{i+1}p_i-2v_{i+1}\int dx[-u_iv_{i+1}+u_iv_i\bar{p}_{i+1}(p_i)_x]'=0$$
$$\dot{v}_i+v''_i-2v_i\bar{p}_{i+1}\bar{p}'_i+2v_i\int dx[v_iu_{i-1}+u_iv_i\bar{p}_{i-1}(p_i)_x]'=0$$
$$-\dot{u}_i+u''_i+2u_i\bar{p}_{i-1}\bar{p}'_i-2u_i\int dx[-u_iv_{i+1}+u_iv_i\bar{p}_{i+1}(p_i)_x]'=0$$
$$-\dot{u}_{i-1}+u''_{i-1}+2u_i\bar{p}'_{i-1}p'_{i-1}-2u_{i-1}\bar{p}_{i-1}p'_i+2v_{i+1}\int dx[v_iu_{i-1}+u_iv_i\bar{p}_{i-1}(p_i)_x]'=0$$
$$\dot{p}_i+p''_i-2(\ln u_i)'p'_i-2u_iv_i\bar{p}'_{i-1}=0 \qquad (29)$$
$$\dot{p}_{i+1}+p''_{i+1}+2(\ln v_i)'p'_{i+1}-2\frac{v_{i+1}}{v_i}\bar{p}'_i=0$$
$$(\theta_i^{-1}\dot{(\bar{p}_i)})_x=p''_{i+1}+p''_{i-1}+2p'_{i+1}(\ln v_i)'+2p'_{i-1}(\ln u_i)'-2\bar{p}'_i(\theta_{i+1}+\theta_{i-1})$$
$$(\theta_{i-1}^{-1}\dot{(\bar{p}_{i-1})})_x=p''_i+p''_{i-2}+2p'_i(\ln v_{i-1})'+2p'_{i-2}(\ln u_{i-1})'-2\bar{p}'_{i-1}(\theta_i+\theta_{i-2})$$

The last two equations are direct consequence of the two previous ones by help of UToda $(2,2)$ transformation may be represent in terms of only unknown functions. So this system is closed, integrable and sequence of its particular solutions is given by formulae

$$p_i=\alpha_i-\nu_i,\quad \bar{p}_i=\bar{\alpha}_i-\bar{\nu}_i,\quad v_i=\frac{[i+1]}{[i]},\quad u_i=\frac{[i+1]}{[i]}$$

as it was shown above.

Now we are able to clarify situation with solution of chain system (27). It is obvious that the system (30) possess particular solution of the form

$$u_0 = u_{-1} = p_0 = \bar{p}_0 = p_{-1} = \bar{p}_{-1} = 0$$

Indeed in the case of final dimensional algebra A_n all matrix elements above are equal to 0. For remaining unknown functions $v_0, v_1, p_1, \bar{p}_1$ as a corollary of (30) we obtain the following (unclosed) system of equations (after trivial regrouping)

$$\dot{v}_0 + v_0'' = V_0 v_0, \quad \dot{(p_1 v_0)} + (p_1 v_0)'' = V_0(p_1 v_), \quad \dot{v}_1 + v_1'' - U_0 v_1 = -2p_1' \bar{p}_1' v_0$$

(the arising of arbitrary functions $(U_, V_0)$ is connected with ambiguity of $\int dx 0 = F(y, t_2))$.

So we see that the functions are $v_0, p_1 v_0)$ are linear independent solutions of Schrodinger equation with arbitrary potential function V_0. While v_1 is the solution unhomoginios Schrodinger equation (with known search function $-2p_1' \bar{p}_1' v_0$ and potential (also arbitrary function) U_0.

In terms of different solutions of tis pair of Schrodiger equations it is possible represent general solution of the chain (27). To this problem we hope to come back in some other place.

5 UToda(m_1, m_2) system and its general solution

In this section we represent some necessary auxiliary relations of representation theory of semisimple algebras. Construction the equations of UToda (m_1, m_2) chains together with its general solution after this takes the form of pure technical manipulations.

Let the pair of operators $m^{\pm}$ satisfy the equations, generalizing (14)

$$m^+_y = m^+(-(h\bar{\Phi})_y + \sum_{s=1}^{q} (Y^s \bar{\phi}^{(s)}) \equiv m^+ L^+$$

$$(Y^s \bar{\phi}^{(s)}) \equiv \sum_i (Y^s_i \bar{\phi}^{(s}_i), \quad (Y^s \phi^{(s)})^T \equiv \sum_i (Y^{-s}_i \phi^{(s}_i) \tag{30}$$

$$m_x^- = m^-((h\Phi)_x - \sum_{s=1}^{q}(Y^s\bar{\phi}^{(s)})^T \equiv m^+L^-$$

where generators $Y^{\pm s}$ together with their commutation relations are defined by (8) and below.

Our nearest goal is to find recurrent relations (or equations) which some matrix elements of the different fundamental representations of the single group element

$$k = m_-^{-1}m_+ \tag{31}$$

satisfy. We have

$$(\ln[i])_{xy} = [i]^{-2}\begin{pmatrix} \langle i \parallel K \parallel i\rangle & \langle i \parallel L^-K \parallel i\rangle \\ \langle i \parallel KL^+ \parallel i\rangle & \langle i \parallel L^-KL^+ \parallel i\rangle \end{pmatrix} \tag{32}$$

In connection with (15) under the action on minimal state vector by the operators of the simple roots only $X_i^+ \parallel i\rangle \neq 0, \langle i \parallel X_i^- \neq 0$. By this reason the action of operators $L^\pm$ on the minimal state vector may be represented in the form

$$L^+ \parallel i\rangle = l_i^+X_i^+ \parallel i\rangle, \quad \langle i \parallel L^- = \langle i \parallel X_i^-l_i^-$$

where $l_i^\pm$ some operators polynomials in generators of positive (negative) simple roots. For instance

$$(Y^2\bar{\phi}^{(2)}) \parallel i\rangle = (\phi_{i-1}^{(2}X_{i-1}^+ - \phi_i^{(2}X_{i+1}^+)X_i^+ \parallel i\rangle$$

or in this case $l_i^+ = \phi_{i-1}^{(2}X_{i-1}^+ - \phi_{i+1}^{(2}X_{i+1}^+$. Keeping this fact in mind and taking into account (8) we can rewrite (32) in the form

$$(\ln[i])_{xy} = [i]^{-2}(l_i^-)_l(l_i^+)_r[i-1][i+1] \tag{33}$$

where now $(l_i^-)_l, (l_i^+)_r$ are the same polynomials constructed from the generators of simple roots correspondingly of the left and right adjoint representations. Formulae below are illustration of application of the last general equalities (33) to concrete cases under the choice $q = 3$ in (30).

$$(\alpha_i)_y = \theta_i(\bar{\phi}_i^{(1} + \bar{\phi}_{i-1}^{(2}\bar{\alpha}_{i-1} - \bar{\phi}_i^{(2}\bar{\alpha}_{i+1} + \bar{\alpha}_{i+1,i+2} - \bar{\alpha}_{i+1}\bar{\alpha}_{i-1} + \bar{\alpha}_{i-1,i-2})$$

$$(\alpha_{i,i+1})_y = \alpha_{i+1}(\alpha_i)_y - \theta_i\theta_{i+1}(\bar{\phi}_i^{(2} + \bar{\alpha}_{i-1} - \bar{\alpha}_{i+2}) \tag{34}$$

$$(\ln[i])_{xy} = \theta_{i+2}\theta_{i+1}\theta_i + \theta_{i+1}\theta_i\theta_{i-1} + \theta_{i-2}\theta_{i-1}\theta_i +$$

$$\theta_{i+1}\theta_i(\phi_i^{(2}+\alpha_{i-1}-\alpha_{i+2})(\bar{....})+\theta_{i-1}\theta_i(\phi_{i-1}^{(2}+\alpha_{i-2}-\alpha_{i+1})(\bar{....})$$
$$\theta_i(\phi_i^{(1}+\phi_{i-1}^{(2}\alpha_{i-1}-\phi_i^{(2}\alpha_{i+1}+\alpha_{i+1,i+2}-\alpha_{i+1}\alpha_{i-1}+\alpha_{i-1,i-2})(\bar{....})$$

where by $(\bar{....})$ we understand the same values as in the first multiplicator in which all functions changed on the bar ones. The same relations as (34) obviously takes place when all functions α are changed on the bar ones together with $y \to x$ and visa versa.

Now let us introduce the new functions $p_i^{(1,2}, \bar{p}_i^{(1,2}$ by the relations

$$p_i^{(1}=\theta_i^1(\bar{\alpha}_i)_y, \quad \bar{p}_i^{(1}=\theta_i^1(\alpha_i)_x$$

$$p_i^{(2}=\phi_i^{(2}+\alpha_{i-1}-\alpha_{i+2}, \quad \bar{p}_i^{(2}=\bar{\phi}_i^{(2}+\bar{\alpha}_{i-1}-\bar{\alpha}_{i+2}$$

Using once more (34) we obtain the chain of equations which functions $p_i^{(1,2}, \bar{p}_i^{(1,2}, \theta_i$ satisfy

$$(p_i^{(2)})_y=\theta_{i-1}\bar{p}_{i-1}^{(1}-\theta_{i+2}\bar{p}_{i+2}^{(1}, \quad (\bar{p}_i^{(2)})_x=\theta_{i-1}p_{i-1}^{(1}-\theta_{i+2}p_{i+2}^{(1}$$

$$(p_i^{(1)})_y=\theta_{i-1}\bar{p}_{i-1}^{(1}p_{i-1}^{(2}-\theta_{i+1}\bar{p}_{i+1}^{(1}p_i^{(2}+\theta_{i-1}\theta_{i-2}\bar{p}_{i-2}^{(2}-\theta_{i+1}\theta_{i+2}\bar{p}_{i+1}^{(2}$$
$$(\bar{p}_i^{(1)})_x=\theta_{i-1}p_{i-1}^{(1}\bar{p}_{i-1}^{(2}-\theta_{i+1}p_{i+1}^{(1}\bar{p}_i^{(2}+\theta_{i-1}\theta_{i-2}p_{i-2}^{(2}-\theta_{i+1}\theta_{i+2}p_{i+1}^{(2}$$
$$(\ln\theta_i)_{xy}=\hat{K}(\theta_i\theta_{i+1}\theta_{i+2}+\theta_i\theta_{i-1}\theta_{i-2}+\theta_i\theta_{i+1}p_i^{(2}\bar{p}_i^{(2}+\theta_{i-1}\theta_i p_{i-1}^{(2}\bar{p}_{i-1}^{(2}+\theta_i p_i^{(1}\bar{p}_i^{(1}$$

This is exactly UToda $(3,3)$ chain system with known general solution which determines by the set of arbitrary functions $(\Phi_i, \phi_i^{(1}, p_i^{(2)})$ and $(\bar{\Phi}_i, \bar{\phi}_i^{(1}, \bar{p}_i^{(2)})$ of single arguments (x,y) correspondingly.

In general case literally repeating calculations of this section or corresponding places of sections 2-3 we come finally to expressions for equtions of integrable UToda $(2,2)$ substitution

$$(p_\alpha^{(s)})_y=\sum_{k=1}^{m_1-s}(p_{\alpha-k}^{(s+k}\bar{p}_{\alpha-k}^{(k}\prod_{i=1}^{k}\theta_{\alpha-k+i-1}-p_\alpha^{(s+k}\bar{p}_{\alpha+s}^{(k}\prod_{i=1}^{k}\theta_{\alpha+s+i-1})=0$$

$$(\ln\theta_\alpha)_{xy}=\hat{K}\sum_{s=1}^{min(m_1,m_2)}\sum_{k=0}^{s-1}p_{\alpha-k}^{(s}\bar{p}_{\alpha-k}^{(s}\prod_{i=1}^{s}\theta_{\alpha-s+i-1}, \quad p_\alpha^{(m_1}=1, \quad \bar{p}_\alpha^{(m_2}=1 \tag{35}$$

$$(\bar{p}_\alpha^{(s)})_y=\sum_{k=1}^{m_2-s}(\bar{p}_{\alpha-k}^{(s+k}p_{\alpha-k}^{(k}\prod_{i=1}^{k}\theta_{\alpha-k+i-1}-\bar{p}_\alpha^{(s+k}p_{\alpha+s}^{(k}\prod_{i=1}^{k}\theta_{\alpha+s+i-1})=0$$

6 Outlook

At first we summarize in few words the construction of the present paper (excluding all details).

In foundation of it are two groups elements $m_\pm$ belonging correspondingly to $\pm$ resolvable subgroups of some semisimple group. They determines by the pair of S-matrix type equations

$$l_+ \equiv m_+^{-1}(m_+)_y = \sum_{s=0}^{m_1} A^{+s}, \quad l_- \equiv m_-^{-1}(m_-)_x = \sum_{s=0}^{m_2} A^{-s} \tag{36}$$

The nature of these equations are pure algebraic - this is condition that lagrangian operators are decomposed on the operators of algebra with graded indexes less then $m_1, (m_2)$.

Matrix elements of different fundamental representations of the single group element $K = m_-^{-1} m_+$ satisfy definite system of equalities which can be interpret as as equations of exactly integrable UToda(m_1, m_2) system.

On this step construction of UToda (m_1, m_2) integrable mapping (substitution) is closed.

Next step is connected with introduction of evolution times parameters. Arbitrary up to now functions $\phi_i^{(s}(y), \phi_i^{(s}(x)$ are restricted by the conditions that elements $m_\pm$ satisfy additional system of equations

$$m_+^{-1}(m_+)_{\bar{t}_{d_1}} = \sum_{s=0}^{d_1} B^{+s}, \quad m_-^{-1}(m_-)_{t_{d_2}} = \sum_{s=0}^{d_2} B^{-s} \tag{37}$$

Condition of selfconsistency of (36) with (37) determines explicit dependence of functions $\phi_i^{(s} \equiv \phi_i^{(s}(y, \bar{t}_1, \bar{t}_2, ...), \phi_i^{(s} \equiv \phi_i^{(s}(x, t_1, t_2, ...)$ on evolution times parameters and space coordinates of the problem x, y.

As a result we obtain the integrable hierarchies of evolution type equations (each system of which determines by different choice of d_1, d_2 under the fixed m_1, m_2) all invariant with respect to UToda (m_1, m_2) substitution.

In one-dimensional case ($\frac{\partial}{\partial y} = \frac{\partial}{\partial y}$) this construction is equivalent to multi time formalism with corresponding technique of Hamiltonians flows [4]

Now we want to enumerate the problems which may be resolved in context of the results of the present paper.

There is no doubts in possibility of the direct (literally) generalization of this construction on supersymmetrical case. As a result it will be possible to obtain unknown up to now integrable hierarchies in (2|2) superspace.

Excellent interesting is the problem of generalization on the quantum domain. Heisenberg operators of usual two-dimentional Toda chain (under canonical rules of quantization) [12] may be expressed as the matrix elements of single quantum group element $k = m_-^{-1} m_+$, where elements of quantum groups $m_\pm$ are the solution of the following system of equations

$$(m_+)_y = m_+(\sum_{s=1}^{r} X_s^+ \exp(k\bar{\phi}(y))_s, \quad (m_-)_x = m_-(\sum_{s=1}^{r} X_s^- \exp(k\phi(x))_s \quad (38)$$

$X_s^\pm$ now are the generators of the simple roots of quantum algebra, $\phi(x) + \bar{\phi}(y)$ is the quantum solution of two dimensional Laplace equation, k-Cartan matrix of semisimle algebra.

Comparison (36) with (38) shows us that the quantum version of UToda chains is not a fantastic suggestion but the problem of the nearest future.

In the present paper we have considered only one example of the system invariant with respect to UToda(2,2) substitution. We hope and shure that solution of symmetry equation in the case of UToda substitutions may be solved by the similar methods as it was done in [3] in the case of the usual Toda chain. But now we are not ready to solve this problem.

And the last comment. Reader can marked the deep disconnection between the simplicity and pure algebraic nature of foundation of construction (only single group element k and equations (36), (37) its define) from one hand and numerous nontrivial recurrent relations between the matrix elements of the different fundamental represention which it is necessary to prove by independent consideration (under approach of the present paper) from the other side. It is possible to hope that the last recurrent relations are the direct corollary of (36), (37) but how to extract from them this information is unknown to the author and may be this is the most interesting problem for the future investigation and the most important output of the present paper.

7 Appendix

Let us consider the function

$$R^i = \langle i \parallel K X_{i-1}^+ X_i^+ \parallel i\rangle[i-1] + \langle i-1 \parallel K X_i^+ X_{i-1}^+ \parallel i-1\rangle[i] -$$
$$\langle i \parallel K X_i^+ \parallel i\rangle\langle i-1 \parallel K X_{i-1}^+ \parallel i-1\rangle$$

where K is arbitrary group element of $SL(n,R)$ group.

From its definition and properties of the minimal state vector $\| i\rangle$ (15) it follows that function R is annihilated by generators of all right positive and left negative simple roots. This means that R by itself is some linear combinations of the matrixes elements taken between the minimal state vectors. Calculations of left (right) Cartan elements shows that they take the definite values on R

$$h_r^s R^i = -(\delta_{s,i+1} + \delta_{s,i-2}) R^i, \quad h_l^s R^i = -(\delta_{s,i} + \delta_{s,i-1}) R^i$$

From the last equalities it follows that from right and left R^i belongs to different irreducible representation. This is impossible and so $R^i = 0$.

Now we enumerate the simplest state vectors of i fundamental representation. They are different by the number of generators of the simple roots applied to the the minimal state vector. Zero order $\| i\rangle$ and first order $X_i^+ \| i\rangle$ states have the dimension one. Second order is two dimensional $X_{i+1}^+ X_i^+ \| i\rangle, X_{i-1}^+ X_i^+ \| i\rangle$. There are three state vectors of the third order $X_{i+2}^+ X_{i+1}^+ X_i^+ \| i\rangle, X_{i+1}^+ X_{i-1}^+ X_i^+ \| i\rangle, X_{i-2}^+ X_{i-1}^+ X_i^+ \| i\rangle$ and fife ones of the fourth order $X_{i+3}^+ X_{i+2}^+ X_{i+1}^+ X_i^+ \| i\rangle, X_{i+2}^+ X_{i-1}^+ X_{i+1}^+ X_i^+ \| i\rangle, X_i^+ X_{i+1}^+ X_{i-1}^+ X_i^+ \| i\rangle, X_{i-2}^+ X_{i-1}^+ X_{i+1}^+ X_i^+ \| i\rangle, X_{i-3}^+ X_{i-2}^+ X_{i-1}^+ X_i^+ \| i\rangle$ All other possibilities give the state vectors with zero norm.

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